Record events in growing populations: Universality, correlation, and aging

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This paper studies the occurrence of record events in score populations which grow stochastically in time. In Rényi's basic record model, a population of independent and identically distributed (i.i.d.) random scores grows deterministically—a single new score being added at each time step. Rényi's record theorem asserts that the resulting record events are independent, and that their occurrence probabilities decrease harmonically in time. Moreover, Rényi's result is universal—being independent of the distribution of the i.i.d. random scores. This paper considers an arbitrary stochastic growth of the score population—allowing the number of the i.i.d. random scores added at each time step to follow arbitrary stochastic dynamics. Exploring the stochastic growth model we: (i) establish a general analog of Rényi's record theorem; (ii) show that universality with respect to the distribution of the i.i.d. random scores is maintained; (iii) compute the distribution of the waiting times for record events; (iv) analyze the dependencies/independencies of the record events; and (v) analyze the aging/stationarity of the record events.

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I. INTRODUCTION

Extreme value theory studies the statistics of extremes [1,2] and is of major importance in the quantitative analysis of rare and catastrophic events such as floods in hydrology, large claims in insurance, crashes in finance, material failure in corrosion analysis, etc [3,4]. One of the topics addressed by extreme value theory is *record events* [5,6]. The study of record events applies to various fields of science—some research examples include: athletics [7], evolution in fitness landscapes [8], climate change [9], and global warming [10].

Consider a sequence of independent and identically distributed (i.i.d.) random scores $\{X(t)\}_{t\geq 1}$ indexed by discrete time ($t=1,2,\cdots$). The random score can represent, for example, the jump a random walker makes at time t. The random score $X(t)$ is said to be a *record* if it is greater than all previous scores $[X(1),\cdots,X(t-1)]$. Let $\mathcal{R}(t)$ denote the record event $\{X(t)$ is a record$\}$. Then, a celebrated theorem by Rényi asserts that [11]: *The record events* $\{\mathcal{R}(t)\}_{t>1}$ *are independent, and their occurrence probabilities decrease harmonically in time*: $\Pr[\mathcal{R}(t)]=1/t$. An alternative representation of Rényi's theorem is given by:

$$\Pr[\mathcal{R}(t_1)\cap\cdots\cap\mathcal{R}(t_k)]=\frac{1}{t_1}\cdots\frac{1}{t_k}. \quad (1)$$

Equation (1) holding for all finite and increasing sequences of integers $1<t_1<\cdots<t_k$.

A remarkable feature of Rényi's theorem is its *universality*: The statistics of the record events $\{\mathcal{R}(t)\}_{t>1}$ are invariant with respect to the distribution of the underlying i.i.d. random scores $\{X(t)\}_{t\geq 1}$. To illustrate the universality of Rényi's theorem consider the aforementioned random walk example: No matter the distribution governing the random walker's jumps—e.g., be they bounded or unbounded, or be they Gaussian or Lévy flights—the temporal statistics of the record jumps are always governed by Eq. (1) and are unaffected by the jumps' distribution.

A straightforward consequence of Rényi's theorem regards the *waiting times* $\{W(t)\}_{t\geq 1}$ for record events. Let $W(t)$ denote the waiting time—from time t onwards—until the occurrence of the first record event after time t. Rényi's theorem implies that the survival probability of the waiting time $W(t)$ is given by:

$$\Pr[W(t)>w]=\frac{t}{t+\lfloor w\rfloor}, \quad (2)$$

($w\geq 0$; $\lfloor w\rfloor$ denoting the integer part of w). Note that the waiting time $W(t)$ is heavy tailed, and has an infinite mean $\langle W(t)\rangle=\infty$. Also note that the *universality* of Rényi's theorem is further induced to the waiting times: The distributions of the waiting times $\{W(t)\}_{t\geq 1}$ are invariant with respect to the distribution of the underlying i.i.d. random scores $\{X(t)\}_{t\geq 1}$.

The statistical model underlying Rényi's theorem—henceforth termed "Rényi's model"—considers the addition of one single score $X(t)$ at every discrete time step ($t=1,2,\cdots$). Rényi's model can be adapted to a continuous-time setting by considering the following Poissonian model [12]: Scores of value x arrive, continuously in time, according to the Poissonian rate $r(x)$. In the Poissonian model the following counterpart of Rényi's theorem holds [12]: The epochs of record arrivals form an inhomogeneous temporal Poisson process with harmonic intensity $\lambda(t)=1/t$ ($t>0$).

Extensions and generalizations of the waiting time result of Eq. (2), in both discrete-time and continuous-time settings, were explored in [13,14]. The research paper [13] considered the Poissonian model, and introduced and explored *geometric waiting times*: given that the current record level is l, and given a parameter $k>1$, how long would we have to wait till the occurrence of a record event whose magnitude is

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at least k times greater than the magnitudes of all the record events preceding it? The research paper [14] considered both Rényi's model and the Poissonian model, and introduced and explored *oligarchy waiting times*: observing the "oligarchy" of the top n scores, how long would we have to wait—from time τ onwards—until a change takes place in this oligarchy? The oligarchy waiting times turned out to establish a universal mechanism for the temporal generation of Paretian power-law distributions with arbitrary integer-valued exponents.

Equations (1) and (2) imply the *aging* of the record events $\{\mathcal{R}(t)\}_{t>1}$: the occurrence of the record events is nonstationary and becomes more and more scarce as time progresses. Indeed, the probability $\Pr[\mathcal{R}(t)]$ decreases harmonically in time, and the median of the waiting time $W(t)$ grows linearly in time $\mathrm{Med}[W(t)] \approx t$. This fact, however, does not comply with our real-life experience (as well as with the yearly updated editions of the Guinness Book of Records). Indeed, in real-life record events are rare—but yet the replacement of old records by new ones is encountered every so often.

One possible reason for the discrepancy between Eqs. (1) and (2) and our real-life experience is *change*. Rényi's model considers the random scores to be identically distributed. In reality, the probability laws of the random scores $\{X(t)\}_{t\geq 1}$ may be time dependent—manifesting the "improvement" of scores as time progresses. Models considering independent, yet not identically distributed, random scores were investigated in [15,16] (see also references therein).

Another possible reason for the discrepancy between Eqs. (1) and (2) and our real-life experience is *growth*. Rényi's model considers the addition of one single score at every time step. In reality however, score populations often grow stochastically and at every time step a random number of scores—rather than a single score—are added. The occurrence of records in score populations which *grow deterministically* (as time progresses) was studied in [17,18]. In this paper we explore the occurrence of records in score populations which *grow stochastically*—their growth governed by *arbitrary stochastic dynamics*.

In the context of the aforementioned random walk example Rényi's model corresponds to tracking the jumps of a single random walker, whereas in this paper we track the jumps of a *growing population* of random walkers. The growth can take place, for example, by the addition of random walkers arriving stochastically in time. Yet another growth example is a population of random walkers which performs a Galton-Watson branching process—each random walker splitting, stochastically in time, into several i.i.d. random walkers.

The manuscript is organized as follows. In Sec. II we present a stochastic growth model for score populations, and present the counterparts of the Rényi's model results—Eqs. (1) and (2). In Sec. III we study the stochastic growth model from a financial perspective—presenting the intrinsic discount rates of the score populations, and using these rates in order to further investigate the occurrence of records in stochastically growing score populations. Regarding the three key features of Rényi's model—*universality*, *independence*, and *aging*—our research concludes that:

(i) The universality with respect to the distribution of the underlying i.i.d. random scores, encountered in Rényi's model, holds also in the general stochastic growth model.

(ii) The independence of the record events encountered in Rényi's model fails, in general, to hold. However, this independence is maintained when the score-population's growth is deterministic and when the score population's intrinsic discount rates form an i.i.d. process.

(iii) The aging of the record events encountered in Rényi's model holds in the general case where the score-population's random score additions form an ergodic process. Stationarity—counterwise to aging—of the record events holds in the general case where the score population's intrinsic discount rates form a stationary process.

Consequently, we asserts that:

(i) When the score-population's intrinsic discount rates form an i.i.d. process then the record events form a *Bernoulli* process—thus maintaining the independence structure of Rényi's model, while yielding stationarity (rather than aging) of the record events.

Throughout the manuscript we consider specific models of stochastic growth: deterministic score additions; ergodic score additions; stationary score additions; ergodic discount rates; stationary discount rates; i.i.d. discount rates.[1] Each of these models is statistically analyzed, and its statistics are compared to the Rényi's model statistics.

II. STOCHASTIC GROWTH MODEL

We consider the following stochastic growth model of the score population: At time t a random number $\Delta(t)$ of i.i.d. random scores is added to the score population; the sequence of score additions $\{\Delta(t)\}_{t\geq 1}$ is an arbitrary random process with arbitrary stochastic dynamics. The size of the score population at time t is thus given by $N(t)=\Delta(1)+\cdots+\Delta(t)$. Note that Rényi's model is a special case of the stochastic growth model with $\Delta(t)\equiv 1$ and $N(t)\equiv t$. Examples of the stochastic growth model include:

(i) Sports records: $\Delta(t)$ being the number of marathon runners having retired at year t; the score of each retiree being her/his best marathon result.

(ii) Flight records: $\Delta(t)$ being the number of aviators having retired at year t; the score of each retiree being her/his flight hours.

(iii) Birth records: $\Delta(t)$ being the number of babies born at year t; the score of each baby being her/his weight at birth.

(iv) Construction records: $\Delta(t)$ being the number of buildings constructed at year t; the score of each building being its constructed area.

(v) Insurance records: $\Delta(t)$ being the number of life-insurance contracts issued at year t; the score of each contract being its value (measured relative to the average income per capita at year t).

In the stochastic growth model we say that a *record event* occurs at time t if one of the scores added at time t is greater than all scores accumulated up to time t (including the scores

[1] Recall that a random process $\{\xi(t)\}_{t=1}^{\infty}$ is ergodic if its temporal averages $\frac{1}{T}\sum_{t=1}^{T} f[\xi(t)]$ converge to a deterministic limit as $T\to\infty$ (the convergence holding, for any test function $f(\cdot)$, with probability one).

added at time t). As above, we let $\mathcal{R}(t)$ denote the record event {a record occurred at time t}. The definition of the *waiting times* $\{W(t)\}_{t\geq 1}$ in the stochastic growth model is as in Rényi's model.

A. Analysis

The counterpart of the Rényi's model results—namely, Eqs. (1) and (2)—are given, respectively, by: (i)

$$\Pr[\mathcal{R}(t_1)\cap\cdots\cap\mathcal{R}(t_k)] = \left\langle \frac{\Delta(t_1)}{N(t_1)}\cdots\frac{\Delta(t_k)}{N(t_k)} \right\rangle, \quad (3)$$

[Eq. (3) holding for all finite and increasing sequences of integers $1<t_1<\cdots<t_k$]; (ii)

$$\Pr[W(t)>w] = \left\langle \frac{N(t)}{N(t+\lfloor w\rfloor)} \right\rangle \quad (4)$$

($w\geq 0$).The proofs of Eqs. (3) and (4) are given in the Appendix. Note that substituting the Rényi model growth—namely, $\Delta(t)\equiv 1$ and $N(t)\equiv t$—into Eqs. (3) and (4) yields back, respectively, the Rényi model results of Eqs. (1) and (2). As in the case of Rényi's model, the results obtained are *universal* with respect to the distribution of the underlying i.i.d. random scores. However—counterwise to Rényi's model—in the stochastic growth model the record events $\{\mathcal{R}(t)\}_{t>1}$ are, in general, *dependent* events.

B. Deterministic score additions

Consider now the *deterministic additions model* in which the score additions $\{\Delta(t)\}_{t\geq 1}$ form a *deterministic* process. This model was introduced in [17] and further explored in [18]. Note that Rényi's model—for which $\Delta(t)\equiv 1$—is a special case of the deterministic additions model. This model retains the independence structure of Rényi's model—rendering the record events $\{\mathcal{R}(t)\}_{t>1}$ *independent*. Three illustrative examples of the deterministic additions model are:

(i) Score populations with power-law growth: $N(t)\approx t^{\alpha}$ (with exponent $\alpha>0$). In this example the record probability $\Pr[\mathcal{R}(t)]$ follows an asymptotically harmonic decrease, and the scaled waiting time $W(t)/t$ is asymptotically Pareto-distributed (with exponent α):

$$\Pr[\mathcal{R}(t)] \approx \frac{\alpha}{t}; \quad \Pr\left[\frac{W(t)}{t}>w\right] \approx \frac{1}{(1+w)^{\alpha}} \quad (5)$$

($t\gg 1$; $w\geq 0$).

(ii) Score populations with stretched-exponential growth: $N(t)\approx\exp(t^{\alpha})$ (with exponent $0<\alpha<1$). In this example the record probability $\Pr[\mathcal{R}(t)]$ follows an asymptotically power-law decrease and the scaled waiting time $W(t)/t^{1-\alpha}$ is asymptotically exponentially distributed (with mean $1/\alpha$):

$$\Pr[\mathcal{R}(t)] \approx \frac{\alpha}{t^{1-\alpha}}; \quad \Pr\left[\frac{W(t)}{t^{1-\alpha}}>w\right] \approx \exp(-\alpha w) \quad (6)$$

($t\gg 1$; $w\geq 0$).

(iii) Score populations with exponential growth: $N(t)\approx\exp(\kappa t)$ (with rate $\kappa>0$). In this example the record probability $\Pr[\mathcal{R}(t)]$ is asymptotically constant, and the waiting time $W(t)$ is asymptotically exponentially distributed (with mean $1/\kappa$):

$$\Pr[\mathcal{R}(t)] \approx 1-\exp(-\kappa); \quad \Pr[W(t)>w] \approx \exp(-\kappa\lfloor w\rfloor) \quad (7)$$

($t\gg 1$; $w\geq 0$). The derivations of Eqs. (5) and (6) are given in the Appendix. Note that in the first two examples—power-law and stretched-exponential growth—*aging* of the record events $\{\mathcal{R}(t)\}_{t>1}$ takes place. Namely, the occurrence of the record events is nonstationary and becomes more and more scarce as time progresses. In the third example however—exponential growth—the record events $\{\mathcal{R}(t)\}_{t>1}$ are asymptotically *stationary*. For growth of the type $N(t)\approx\exp(t^{\alpha})$ further note the dramatic phase transition taking place as the exponent α changes from $\alpha<1$ (example 2) to $\alpha=1$ (example 3).

C. Ergodic and stationary score additions

Consider now the *ergodic additions model* in which the random score additions $\{\Delta(t)\}_{t\geq 1}$ form an *ergodic* process with ergodic average $\bar{\Delta}=\lim_{T\to\infty}\frac{1}{T}\Sigma_{t=1}^{T}\Delta(t)$. Note that Rényi's model—for which $\Delta(t)\equiv 1$—is a special case of the ergodic additions model (with ergodic average $\bar{\Delta}=1$).

In the ergodic additions model the record events $\{\mathcal{R}(t)\}_{t>1}$ are, in general, *dependent* events. Yet, the *aging* encountered in Rényi's model holds also in the ergodic additions model. Indeed, Eqs. (3) and (4)—combined together with the ergodicity of the random process $\{\Delta(t)\}_{t\geq 1}$—imply that: (i) the record probability $\Pr[\mathcal{R}(t)]$ follows an asymptotically harmonic decrease:

$$\Pr[\mathcal{R}(t)] \approx \frac{\langle\Delta(t)\rangle}{\bar{\Delta}}\frac{1}{t}, \quad (8)$$

($t\gg 1$); (ii) the scaled waiting time $W(t)/t$ is asymptotically Pareto-distributed (with exponent $\alpha=1$):

$$\Pr\left[\frac{W(t)}{t}>w\right] \approx \frac{1}{1+w} \quad (9)$$

($t\gg 1$; $w\geq 0$). The derivations of Eqs. (8) and (9) are given in the Appendix. If the random score additions $\{\Delta(t)\}_{t\geq 1}$ form a *stationary* process then stationarity further implies that the ratio $\langle\Delta(t)\rangle/\bar{\Delta}$ appearing in Eq. (8) equals unity. (Recall that the feature distinguishing between ergodicity and stationarity is periodicity. Namely, ergodic processes can be periodic, while stationary processes cannot. The simplest example of a nonstationary ergodic process is the alternating sequence $\{0,1,0,1,\cdots\}$.)

III. INTRINSIC DISCOUNT RATES

The deterministic examples presented in Sec. II B demonstrated two dramatically different statistical behaviors the record events $\{\mathcal{R}(t)\}_{t>1}$ can exhibit: *aging* and *stationarity*. In Sec. II C we further concluded that the general case of ergodic additions always leads to aging of the record events. On the other hand, in Sec. II B we saw that deterministic exponential growth leads to stationarity of the record events. Is the stationarity exhibited by deterministic exponential growth unique—or is it shared by other stochastic growth models? In order to answer this question we need explore the geometric structure of the stochastic growth, and to that end we now turn to analyze the growth dynamics from a financial perspective.

Consider the random process $\{N(t)\}_{t\geq 1}$. The process's *intrinsic interest rate* $I(t)$ at time $t>1$ is given by: $N(t)=[1+I(t)]N(t-1)$. And, the process's *intrinsic discount rate* $D(t)$ at time $t>1$ is given by: $D(t)=1/[1+I(t)]$. Namely, the score population's intrinsic discount rates $\{D(t)\}_{t>1}$ are given by the ratios $D(t)=N(t-1)/N(t)$ (note that these ratios take values in the unit interval).

In terms of the intrinsic discount rates $\{D(t)\}_{t>1}$ Eqs. (3) and (4) admit, respectively, the following representations: (i)

$$\Pr[\mathcal{R}(t_1) \cap \cdots \cap \mathcal{R}(t_k)] = \langle [1-D(t_1)] \cdots [1-D(t_k)] \rangle, \tag{10}$$

[Eq. (10) holding for all finite and increasing sequences of integers $1<t_1<\cdots<t_k$]; (ii)

$$\Pr[W(t) > w] = \langle D(t+1) \cdots D(t+\lfloor w \rfloor) \rangle, \tag{11}$$

($w\geq 0$). The derivations of Eqs. (10) and (11) are given in the Appendix.

An immediate implication of the intrinsic discount rates representation regards the *correlation structure* of the record events $\{\mathcal{R}(t)\}_{t>1}$. Indeed, Eq. (10) implies that the *autocorrelation* function of the record events $\{\mathcal{R}(t)\}_{t>1}$ coincides with the *autocorrelation* function of the intrinsic discount rates $\{D(t)\}_{t>1}$:

$$\boldsymbol{\rho}[\mathcal{R}(t),\mathcal{R}(s)] = \boldsymbol{\rho}[D(t),D(s)], \tag{12}$$

[Eq. (12) holding for all integers $t,s>1$]. The derivation of Eq. (12) is given in the Appendix.

A. Ergodic discount rates

Consider now the *ergodic discount rates model* in which the intrinsic discount rates $\{D(t)\}_{t>1}$ form an *ergodic* process with ergodic average $\bar{D}=\lim_{T\to\infty}\frac{1}{T}\sum_{t=2}^{T+1}D(t)$. Note that the "log-returns" $\{L(t)\}_{t>1}$ of the random process $\{N(t)\}_{t\geq 1}$ are given by $L(t)=\ln[N(t)/N(t-1)]=-\ln[D(t)]$ and let $\bar{L}$ denote the ergodic average of the log-returns: $\bar{L}=\lim_{T\to\infty}\frac{1}{T}\sum_{t=2}^{T+1}L(t)$.

The ergodic discount rates model exhibits a markedly different statistical behavior than Rényi's model—yielding *asymptotic stationarity*, rather than aging, of the record events $\{\mathcal{R}(t)\}_{t>1}$. Indeed, Eqs. (10) and (11)—combined together with the ergodicity of the random process $\{D(t)\}_{t>1}$—imply that: (i) the long-term average of the record probabilities $\{\Pr[\mathcal{R}(t)]\}_{t>1}$ and the ergodic average $\bar{D}$ of the intrinsic discount rates $\{D(t)\}_{t>1}$ sum up to unity:

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}\Pr[\mathcal{R}(t+k)] = 1-\bar{D}, \tag{13}$$

[Eq. (13) holding for all integers $t>1$]. (ii) The waiting times $\{W(t)\}_{t\geq 1}$ are asymptotically exponentially distributed (with mean $1/\bar{L}$):

$$\Pr[W(t) > w] \approx \exp(-\bar{L}w), \tag{14}$$

[Eq. (14) holding for all integers $t\geq 1$; $w\gg 1$]. The derivations of Eqs. (13) and (14) are given in the Appendix.

B. Stationary discount rates

Consider now the *stationary discount rates model* in which the intrinsic discount rates $\{D(t)\}_{t>1}$ form a *stationary* process. As in the case of the ergodic discount rates model, this model exhibits a markedly different statistical behavior than Rényi's model—yielding *stationarity*, rather than aging, of the record events $\{\mathcal{R}(t)\}_{t>1}$. Indeed, Eqs. (10) and (11)—combined together with the stationarity of the random process $\{D(t)\}_{t>1}$—imply that: (i) the record events $\{\mathcal{R}(t)\}_{t>1}$ form a *stationary* sequence of random events—with *constant* occurrence probability

$$\Pr[\mathcal{R}(t)] = 1-\langle D(2)\rangle. \tag{15}$$

(ii) The waiting times $\{W(t)\}_{t\geq 1}$ form a *stationary* sequence of random variables—their probability law governed by the temporally invariant survival probability

$$\Pr[W(t) > w] = \langle D(2) \cdots D(1+\lfloor w \rfloor) \rangle \tag{16}$$

($w\geq 0$).

C. I.i.d. discount rates

Finally, consider the *i.i.d. discount rates model* in which the intrinsic discount rates $\{D(t)\}_{t>1}$ form an i.i.d. process—i.e., a sequence of i.i.d. random variables—with generic random value D. In this model the record events $\{\mathcal{R}(t)\}_{t>1}$ form a *Bernoulli* process. Indeed, Eqs. (10) and (11)—combined together with the i.i.d. structure of the random process $\{D(t)\}_{t>1}$—imply that: (i) The record events $\{\mathcal{R}(t)\}_{t>1}$ are *i.i.d.* random events—with *constant* occurrence probability

$$\Pr[\mathcal{R}(t)] = 1-\langle D\rangle. \tag{17}$$

(ii) The waiting times $\{W(t)\}_{t\geq 1}$ form a *stationary* sequence of *geometrically distributed* random variables—their probability law governed by the geometric survival probability

$$\Pr[W(t) > w] = \langle D\rangle^{\lfloor w \rfloor} \tag{18}$$

($w\geq 0$). Counterwise to Rényi's model, in the i.i.d. discount rates model: (i) the occurrence probabilities of the record events $\Pr[\mathcal{R}(t)]$ are constant in time—rather than decreasing harmonically in time; (ii) the waiting times $\{W(t)\}_{t\geq 1}$ are temporally invariant and geometrically distributed—rather than time dependent and heavy tailed. On the other hand, both the i.i.d. discount rates model and Rényi's model share the common feature of independent record events. Hence, the i.i.d. discount rates model retains the independence structure of Rényi's model, while yielding stationarity—rather than aging—of the record events $\{\mathcal{R}(t)\}_{t>1}$.

IV. CONCLUSIONS

This research paper embarked from Rényi's celebrated record theorem. Rényi's model was adapted and generalized so that to accommodate the—rather prevalent—case of stochastically growing score populations. The statistics of record events—in score populations whose growth follows arbitrary stochastic dynamics—were comprehensively explored and analyzed:

(i) A general analog of Rényi's record theorem, as well as a general formula for the distributions of the waiting times for record events, were established. Both these results maintain the universality of Rényi's model—i.e., they are invariant with respect to the distribution of the underlying i.i.d. random scores.

(ii) Counterwise to Rényi's model, in the stochastic growth model the record events were shown to be dependent events—their correlation structure coinciding with the correlation structure of the score population's intrinsic discount rates. It was further shown that the independence structure of Rényi's model is maintained when the score additions form a deterministic process, and when the intrinsic discount rates form an i.i.d. process.

(iii) Analogous to Rényi's model, aging of the record events was shown to hold in the general case where the score additions form an ergodic process. Counterwise to Rényi's model, asymptotic stationarity (stationarity) of the record events was shown to hold in the general case where the intrinsic discount rates form an ergodic (stationary) process.

(iv) The case of i.i.d. intrinsic discount rates was shown to render the record events a Bernoulli process—hence maintaining the independence structure of Rényi's model, but yet yielding stationarity of the record events.

The results obtained provide a general statistical machinery for the quantitative analysis and prediction of the occurrence of record events in diverse fields of science—where the underlying score populations grow stochastically.

APPENDIX

Throughout the Appendix: (i) $f(x)$ and $F(x)$ denote, respectively, the probability density function and the cumulative distribution function of the i.i.d. random scores; (ii) we set $t_0 := 0$ and $N(t_0) := 0$; (iii) $\mathbf{E}[\cdot]$ denotes mathematical expectation.

1. Proof of Eq. (3)

Let $E(t;dx)$ denote the event {a record of magnitude $x \in (x, x+dx)$ took place at time t}. For $1 < t_1 < \cdots < t_k$ and $x_1 < \cdots < x_k$ we have

$$\Pr[E(t_1;dx_1) \cap \cdots \cap E(t_k;dx_k)|\Delta] = \prod_{j=1}^{k} \underbrace{F(x_j)^{N(t_j-1)-N(t_{j-1})}}_{\text{I}} \cdot \underbrace{\Delta(t_j)F(x_j)^{\Delta(t_j)-1}f(x_j)dx_j}_{\text{II}} = \prod_{j=1}^{k} \Delta(t_j)F(x_j)^{N(t_j)-N(t_{j-1})-1}f(x_j)dx_j, \tag{A1}$$

[part I is the probability that all scores arriving at times $t_{j-1}+1, t_{j-1}+2, \cdots, t_j-1$ are smaller than x_j; part II is the probability that among all scores arriving at time t_j one is of magnitude $x \in (x_j, x_j+dx_j)$, and all other are smaller than x_j]. Now:

$$\Pr[\mathcal{R}_{t_1} \cap \cdots \cap \mathcal{R}_{t_k}|\Delta] = \int \cdots \int_{x_1<\cdots<x_k} \Pr[E(t_1;dx_1) \cap \cdots \cap \times E(t_k;dx_k)|\Delta], \tag{A2}$$

[using Eq. (A1)]

$$= \int \cdots \int_{x_1<\cdots<x_k} \left(\prod_{j=1}^{k} \Delta(t_j)F(x_j)^{N(t_j)-N(t_{j-1})-1}f(x_j)dx_j \right), \tag{A3}$$

[using the change in variables $u_1 = F(x_1), \cdots, u_k = F(x_k)$]

$$= \int \cdots \int_{0<u_1<\cdots<u_k<1} \left(\prod_{j=1}^{k} \Delta(t_j)u_j^{N(t_j)-N(t_{j-1})-1}du_j \right), \tag{A4}$$

(integrating the variable u_1)

$$= \frac{\Delta(t_1)}{N(t_1)} \int \cdots \int_{0<u_2<\cdots<u_k<1} \left(u_2^{N(t_1)} \prod_{j=2}^{k} \Delta(t_j)u_j^{N(t_j)-N(t_{j-1})-1}du_j \right), \tag{A5}$$

(integrating the variable u_2)

$$= \frac{\Delta(t_1)}{N(t_1)}\frac{\Delta(t_2)}{N(t_2)} \int \cdots \int_{0<u_3<\cdots<u_k<1} \times \left(u_3^{N(t_2)} \prod_{j=3}^{k} \Delta(t_j)u_j^{N(t_j)-N(t_{j-1})-1}du_j \right), \tag{A6}$$

(continuing by induction)

$$= \frac{\Delta(t_1)}{N(t_1)} \cdots \frac{\Delta(t_{k-1})}{N(t_{k-1})} \int_{0<u_k<1} (u_k^{N(t_{k-1})}\Delta(t_k)u_k^{N(t_k)-N(t_{k-1})-1}du_k), \tag{A7}$$

(integrating the variable u_k)

$$= \frac{\Delta(t_1)}{N(t_1)} \cdots \frac{\Delta(t_{k-1})}{N(t_{k-1})}\frac{\Delta(t_k)}{N(t_k)}. \tag{A8}$$

Finally, using probabilistic conditioning, we conclude that:

$$\Pr[\mathcal{R}_{t_1} \cap \cdots \cap \mathcal{R}_{t_k}] = \mathbf{E}\{\Pr[\mathcal{R}_{t_1} \cap \cdots \cap \mathcal{R}_{t_k}|\Delta]\} = \mathbf{E}\left[\frac{\Delta(t_1)}{N(t_1)} \cdots \frac{\Delta(t_k)}{N(t_k)} \right]. \tag{A9}$$

2. Proof of Eq. (4)

Let M_a^b denote the maximal score attained by scores added at times $a, a+1, \cdots, b$. Then:

$$\Pr[M_a^b \le x|\Delta] = F(x)^{N(b)-N(a-1)}, \quad (A10)$$

and the corresponding probability density function is given by

$$f_{M_a^b|\Delta}(x) = [N(b)-N(a-1)]F(x)^{N(b)-N(a-1)-1}f(x). \quad (A11)$$

Now:

$$\Pr[W_t > w|\Delta] = \Pr[W_t > \lfloor w \rfloor|\Delta], \quad (A12)$$

(using probabilistic conditioning)

$$\mathbf{E}[\Pr(W_t > \lfloor w \rfloor|\Delta, M_1^t)] = \mathbf{E}[\Pr(M_{t+1}^{t+\lfloor w \rfloor} < M_1^t|\Delta, M_1^t)], \quad (A13)$$

[using Eq. (A10)]

$$= \mathbf{E}[F(M_1^t)^{N(t+\lfloor w \rfloor)-N(t)}|\Delta], \quad (A14)$$

[using Eq. (A11)]

$$= \int_{-\infty}^{\infty} F(x)^{N(t+\lfloor w \rfloor)-N(t)} \cdot [N(t)F(x)^{N(t)-1}f(x)]dx$$

$$= N(t)\int_{-\infty}^{\infty} F(x)^{N(t+\lfloor w \rfloor)-1}f(x)dx, \quad (A15)$$

[using the change in variables $u = F(x)$]

$$= N(t)\int_0^1 u^{N(t+\lfloor w \rfloor)-1}du = \frac{N(t)}{N(t+\lfloor w \rfloor)}. \quad (A16)$$

Finally, using probabilistic conditioning, we conclude that:

$$\Pr[W_t > \lfloor w \rfloor] = \mathbf{E}[\Pr(W_t > \lfloor w \rfloor|\Delta)] = \mathbf{E}\left[\frac{N(t)}{N(t+\lfloor w \rfloor)}\right]. \quad (A17)$$

3. Derivation of Eqs. (5)–(14)

Eq. (5). Substituting $N(t) \approx t^\alpha$ into Eqs. (3) and (4) yields

$$\Pr[\mathcal{R}(t)] \approx \frac{t^\alpha - (t-1)^\alpha}{t^\alpha} = 1 - \left(1 - \frac{1}{t}\right)^\alpha \approx \frac{\alpha}{t}, \quad (A18)$$

and

$$\Pr\left[\frac{W(t)}{t} > w\right] \approx \frac{t^\alpha}{(t+\lfloor tw \rfloor)^\alpha} = \frac{1}{(1+\lfloor tw \rfloor/t)^\alpha} = \frac{1}{(1+w)^\alpha}. \quad (A19)$$

Eq. (6). Substituting $N(t) \approx \exp(t^\alpha)$ into Eqs. (3) and (4) yields

$$\Pr[\mathcal{R}(t)] \approx \frac{\exp(t^\alpha) - \exp((t-1)^\alpha)}{\exp(t^\alpha)} = 1 - \exp\left(-t^\alpha\left[1 - \left(1 - \frac{1}{t}\right)^\alpha\right]\right) \approx 1 - \exp\left(-t^\alpha\frac{\alpha}{t}\right) \approx \frac{\alpha}{t^{1-\alpha}},$$

and

$$\Pr\left[\frac{W(t)}{t^{1-\alpha}} > w\right] \approx \frac{\exp(t^\alpha)}{\exp((t+\lfloor t^{1-\alpha}w \rfloor)^\alpha)} = \exp\left(-t^\alpha\left[\left(1 + \frac{\lfloor t^{1-\alpha}w \rfloor}{t^{1-\alpha}}\frac{1}{t^\alpha}\right)^\alpha - 1\right]\right) \approx \exp\left(-t^\alpha\left[\left(1+\frac{w}{t^\alpha}\right)^\alpha - 1\right]\right) \approx \exp(-\alpha w). \quad (A20)$$

Eq. (7). Substituting $N(t) \approx \exp(\kappa t)$ into Eqs. (3) and (4) yields

$$\Pr[\mathcal{R}(t)] \approx \frac{\exp(\kappa t) - \exp[\kappa(t-1)]}{\exp(\kappa t)} = 1 - \exp(-\kappa), \quad (A21)$$

and

$$\Pr[W(t) > w] \approx \frac{\exp(\kappa t)}{\exp[\kappa(t+\lfloor w \rfloor)]} = \exp(-\kappa\lfloor w \rfloor). \quad (A22)$$

Eq. (8). From Eq. (3) we obtain that

$$\Pr[\mathcal{R}(t)] = \mathbf{E}\left[\frac{\Delta(t)}{N(t)}\right] = \mathbf{E}\left[\frac{\Delta(t)}{\Delta(1)+\cdots+\Delta(t)}\right] = \mathbf{E}\left[\frac{\Delta(t)}{\frac{\Delta(1)+\cdots+\Delta(t)}{t}}\right]\frac{1}{t} \approx \mathbf{E}\left[\frac{\Delta(t)}{\bar{\Delta}}\right]\frac{1}{t} = \frac{\mathbf{E}[\Delta(t)]}{\bar{\Delta}}\frac{1}{t}. \quad (A23)$$

Eq. (9). From Eq. (4) we obtain that

$$\Pr\left[\frac{W(t)}{t} > w\right] = \mathbf{E}\left[\frac{N(t)}{N(t+\lfloor tw \rfloor)}\right] = \mathbf{E}\left[\frac{\Delta(1)+\cdots+\Delta(t)}{\Delta(1)+\cdots+\Delta(t+\lfloor tw \rfloor)}\right] = \mathbf{E}\left[\frac{\frac{\Delta(1)+\cdots+\Delta(t)}{t}}{\frac{\Delta(1)+\cdots+\Delta(t+\lfloor tw \rfloor)}{t+\lfloor tw \rfloor}}\right]\frac{t}{t+\lfloor tw \rfloor} \approx \mathbf{E}\left[\frac{\bar{\Delta}}{\bar{\Delta}}\right]\frac{1}{1+\frac{\lfloor tw \rfloor}{t}} \approx \frac{1}{1+w}. \quad \text{(A24)}$$

Eq. (10). Using the relation $D(t)=N(t-1)/N(t)$ we obtain that

$$\frac{\Delta(t)}{N(t)} = \frac{N(t)-N(t-1)}{N(t)} = 1-D(t). \quad \text{(A25)}$$

Substituting Eq. (A25) into Eq. (3) yields Eq. (10).

Eq. (11). Using the relation $D(t)=N(t-1)/N(t)$ we obtain that

$$\frac{N(t)}{N(t+\lfloor w \rfloor)} = \frac{N(t)}{N(t+1)}\frac{N(t+1)}{N(t+2)}\cdots\frac{N(t+\lfloor w \rfloor-1)}{N(t+\lfloor w \rfloor)} = D(t+1)D(t+2)\cdots D(t+\lfloor w \rfloor). \quad \text{(A26)}$$

Substituting Eq. (A26) into Eq. (4) yields Eq. (11).

Eq. (12). Using Eq. (10), the covariance between the record events $\mathcal{R}(t)$ and $\mathcal{R}(s)$ is given by

$$\mathbf{Cov}[\mathcal{R}(t),\mathcal{R}(s)] = \Pr[\mathcal{R}(t)\cap\mathcal{R}(s)] - \Pr[\mathcal{R}(t)]\Pr[\mathcal{R}(s)] = \mathbf{E}\{[1-D(t)][1-D(s)]\} - \mathbf{E}\{[1-D(t)]\} \times\mathbf{E}\{[1-D(s)]\} = \mathbf{Cov}[D(t),D(s)]. \quad \text{(A27)}$$

Consequently, the correlation between the record events $\mathcal{R}(t)$ and $\mathcal{R}(s)$ is given by

$$\boldsymbol{\rho}[\mathcal{R}(t),\mathcal{R}(s)] = \frac{\mathbf{Cov}[\mathcal{R}(t),\mathcal{R}(s)]}{\sqrt{\mathbf{Cov}[\mathcal{R}(t),\mathcal{R}(t)]}\sqrt{\mathbf{Cov}[\mathcal{R}(s),\mathcal{R}(s)]}} = \frac{\mathbf{Cov}[D(t),D(s)]}{\sqrt{\mathbf{Cov}[D(t),D(t)]}\sqrt{\mathbf{Cov}[D(s),D(s)]}} = \boldsymbol{\rho}[D(t),D(s)]. \quad \text{(A28)}$$

Eq. (13). From Eq. (10) we obtain that

$$\frac{1}{T}\sum_{k=1}^{T}\Pr[\mathcal{R}(t+k)] = \frac{1}{T}\sum_{k=1}^{T}\mathbf{E}[1-D(t+k)] = 1-\mathbf{E}\left[\frac{1}{T}\sum_{k=1}^{T}D(t+k)\right]. \quad \text{(A29)}$$

Ergodicity further implies that $\lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{T}D(t+k)=\bar{D}$, and hence Eq. (13) follows.

Eq. (14). From Eq. (11) we obtain that

$$\Pr[W(t) > w] = \mathbf{E}\{\exp[-L(t+1)]\cdots\exp[-L(t+\lfloor w \rfloor)]\} = \mathbf{E}\left[\sum_{k=1}^{\lfloor w \rfloor}L(t+k)\right]. \quad \text{(A30)}$$

Ergodicity further implies that $\lim_{w\to\infty}\frac{1}{w}\sum_{k=1}^{\lfloor w \rfloor}L(t+k)=\bar{L}$, and hence Eq. (14) follows.

[1] E. J. Gumbel, *Statistics of Extremes* (Columbia University Press, New York, 1958).

[2] J. Galambos, *Asymptotic Theory of Extreme Order Statistics*, 2nd ed. (Krieger, New York, 1987).

[3] P. Embrechts, C. Kluppelberg, and T. Mikosch, *Modelling Extremal Events for Insurance and Finance* (Springer, New York, 1997).

[4] M. Thomas and R. D. Reiss, *Statistical Analysis of Extreme Values* (Birkhauser, New York, 2001).

[5] B. C. Arnold, N. Balakrishnan, and H. N. Nagaraja, *Records* (Wiley, New York, 1998).

[6] V. B. Nevzorov, *Records* (American Mathematical Society, Providence, 2001).

[7] D. Gembris, J. G. Taylor, and D. Suter, Nature (London) **417**, 506 (2002).

[8] K. Jain and J. Krug, J. Stat. Mech.: Theory Exp. (2005) P04008.

[9] R. E. Benestad, Clim. Res. **25**, 3 (2003).

[10] S. Redner and M. R. Petersen, Phys. Rev. E **74**, 061114 (2006).

[11] A. Rényi, *On Outstanding Values of a Sequence of Observations* Akadémiai Kiadó, Budapest,1962); *Selected Papers of A. Rényi* (Akadémiai Kiadó, Budapest, 1976) Vol. 3.

[12] I. Eliazar, Probl. Eng. Inform. Sci. **19**, 289 (2005).

[13] I. Eliazar, Physica A **348**, 181 (2005).

[14] I. Eliazar and J. Klafter, Physica A **377**, 53 (2007).

[15] K. Borovkov, J. Appl. Probab. **36**, 668 (1999).

[16] J. Krug, J. Stat. Mech.: Theory Exp. (2007) P07001.

[17] M. C. K. Yang, J. Appl. Probab. **12**, 148 (1975).

[18] V. B. Nevzorov, Theory Probab. Appl. **32**, 201 (1987).